Anomalous scaling in the *N*-point functions of a passive scalar

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A recent analysis of the four-point correlation function of the passive scalar advected by a time-decorrelated random flow is extended to the N-point case. It is shown that all stationary-state inertial-range correlations are dominated by homogeneous zero modes of singular operators describing their evolution. We compute analytically the zero modes governing the N-point structure functions and the anomalous dimensions corresponding to them to the linear order in the scaling exponent of the two-point function of the advecting velocity field. The implications of these calculations for the dissipation correlations are discussed. [S1063-651X(96)05808-4]

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I. INTRODUCTION

There has been much effort lately to understand the behavior of a scalar quantity passively advected by a random flow with a Gaussian statistics decorrelated in time [1]. This simple model, of its own interest, has served as a prototype of a turbulent system. It is believed that its behavior may teach us important lessons about the fully developed hydrodynamical turbulence. One of the interesting aspects of the passive scalar which has been recently understood [2–4] is the origin of the breakdown of Kolmogorov inertial-range scaling in the higher structure functions of the scalar. It has been realized that the dominant contribution to the structure functions comes from the zero modes of the differential operators describing the stochastic evolution of the correlation functions of the scalar. In this paper we extend the results of Ref. [2] by presenting the computation of the anomalous dimensions of the N-point structure functions in the first order of the parameter ξ. Exponent ξ, which in [2] was denoted κ and in [3] $2-\gamma$, is the growth rate of the two-point structure function of the velocities of the advecting flow. The present work was motivated by [5] where a similar analysis in the first order in inverse dimension was sketched.

The equation governing the passive scalar in a turbulent flow is

$$\partial_t T+(\mathbf{u}\cdot\nabla)T-\nu\Delta T=f. \tag{1}$$

Here $T(x,t)$ describes the scalar, e.g., the temperature, and f the forcing term whose role is to compensate the dissipation caused by the term proportional to the molecular diffusivity ν. The velocity field $\mathbf{u}$ with $\nabla\cdot\mathbf{u}=0$ is supposed to be random. We shall work in $d\geqslant 3$ space dimensions and shall assume homogeneity, isotropy, and parity invariance of the advecting flow and of the forcing.

The statistics of the forcing term is assumed to be Gaussian with mean zero and two-point function

$$\langle f(x,t)f(y,t')\rangle=C\left(\frac{x-y}{L}\right)\delta(t-t'). \tag{2}$$

The rotation-invariant function $C(x/L)$, which could be chosen to be a Gaussian, varies on scale L.

The statistics of the velocity field, independent of the forcing, is also supposed to be Gaussian with zero mean and with the two-point functions

$$\langle u^\alpha(x,t)u^\beta(y,t')\rangle=D^{\alpha\beta}(x-y)\,\delta(t-t'),\qquad \text{with }\ \partial_\alpha D^{\alpha\beta}=0. \tag{3}$$

To analyze the scaling property of the scalar correlation functions we shall use the following expression for $D^{\alpha\beta}$: $D^{\alpha\beta}(x)=D(0)\,\delta^{\alpha\beta}-d^{\alpha\beta}(x)$, with

$$d^{\alpha\beta}(x)=D\left((d+\xi-1)\,\delta^{\alpha\beta}-\xi\frac{x^\alpha x^\beta}{|x|^2}\right)|x|^\xi, \tag{4}$$

where ξ is a parameter, $0<\xi<2$, see [2] for a description of the origin of this expression. Clearly, the above distribution for u is far from realistic. It mimics, however, the growth of the correlations of velocity differences with separation distance, typical for turbulent flows. The fact that the two-point functions (2) and (3) are white noise in time is crucial for the solvability of the model. The parameter ξ fixes the naive dimensions under the rescalings $x\to\mu x$, $L\to\mu L$. The naive dimension of u is $\xi/2$ and of T is $(2-\xi)/2$. Scale L serves as an infrared cutoff and the "Kolmogorov scale" $\eta=(\nu/D)^{1/\xi}$ as an ultraviolet cutoff.

We shall be interested in the correlation functions of the scalar in the inertial range $\eta\ll x\ll L$. The main result of this paper is that in this range the stationary-state, equal-time, even structure functions scale with the anomalous exponents ρ_N as

$$\langle[T(x,t)-T(0,t)]^N\rangle\cong a_N\left(\frac{L}{|x|}\right)^{\rho_N}|x|^{(2-\xi)N/2}, \tag{5}$$

with

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$$\rho_N = \xi \frac{N(N-2)}{2(d+2)} + O(\xi^2). \tag{6}$$

The exponents are universal depending only on ξ but the amplitudes a_N are not: they depend on the shape of the covariance C. The error term is bounded by $O((L/|x|)^{-2+O(\xi)}|x|^{(2-\xi)N/2})$ so it is strongly suppressed for large $L/|x|$. As it should be, the ρ_N's satisfy the Hölder inequality $\rho_N \geqslant [(N-2)/2]\rho_4$. More precise descriptions and statements will be given below. The formula (6) agrees with the $N=4$ result of [2] and with the $1/d$ expansion of [3,5].

Following Ref. [2], we shall derive the values of the anomalous exponents by analyzing in perturbation theory in ξ the zero modes of differential operators characterizing the stationary state. Although for $\xi=0$ one observes a purely diffusive behavior of T and for $\xi>0$ an inertial energy cascade, the zero modes differ little in both cases. Their behavior in ξ may be controlled by perturbation theory of singular elliptic operators with discrete spectrum. Different physics arises from the cumulative effect of the zero modes. As already stressed in [2], this resembles the situation in the renormalization group analysis in field theory or statistical mechanics where relevant perturbations, controllable in the single scale problem, may have large effects on the behavior of the system. As in the renormalization group study of critical models, the first order perturbative corrections to the zero modes lead to the resummation of leading infrared logarithms in the perturbation expansion of the structure functions in powers of ξ. Pursuing the analogy further, we shall introduce, as in the perturbative renormalization group, the notion of matrix of anomalous dimensions, see also [5].

Our results can be used to deduce the scaling properties of the correlation functions of the dissipation field, which we denote by $\epsilon(x)$, as discussed, for example, in [3,6]. At finite diffusivity $\nu \neq 0$, the dissipation field is defined (inside correlators) by $\epsilon(x) = \nu \lim_{x' \to x} (\nabla T)(x') \cdot (\nabla T)(x)$. This is a sensible definition since at finite ν the correlations of T and their first derivatives are not singular at coinciding points (the higher derivatives are). In the limit $\nu \to 0$, we have alternative definitions:

$$\epsilon(x) = \lim_{\nu \to 0} \nu \lim_{x' \to x} (\nabla T)(x') \cdot (\nabla T)(x) \tag{7}$$

or

$$\epsilon(x) = \lim_{x' \to x} \tfrac{1}{2} [d^{\alpha\beta}(x-x') \partial_{x^\alpha} \partial_{x'^\beta}] \lim_{\nu \to 0} T(x') T(x). \tag{8}$$

The order of the limits in the first definition is crucial since when $\nu \to 0$ and for small $|x-x'|$, $T(x)T(x') \sim |x-x'|^{2-\xi}$ modulo more regular terms so that $(\nabla T)(x) \cdot (\nabla T)(x') \sim |x-x'|^{-\xi}$ and becomes singular. The noncommutativity of the limits $\nu \to 0$ and $x' \to x$ is at the origin of the dissipative anomaly. The second definition of $\epsilon(x)$ is in the spirit of the operator product expansion in the $\nu=0$ theory. Using the Hopf identities (10) for the correlation functions, we shall argue that both expressions for the dissipation field $\epsilon(x)$ coincide for $\xi<1$. The mean dissipation $\bar{\epsilon} \equiv \langle \epsilon(x) \rangle$ is equal to $\frac{1}{2} C(0)$, i.e., to the mean injection rate of energy. The dissipation field has zero naive scaling dimension since $T^2(x)$ and $d(x)\nabla_x^2$ have opposite naive dimensions. However, as a consequence of the relation (8), one finds that $\epsilon(x)$ acquires an anomalous scaling. In fact, the definition (8) and Eq. (16) allow one to compute any structure functions with (noncoincident) insertions of the dissipation field. For example, the connected two-point function of ϵ scales as

$$\langle \epsilon(x), \epsilon(0) \rangle^c \sim \left(\frac{L}{|x|} \right)^{\rho_4} \tag{9}$$

and it decreases with $|x|$, in agreement with the physical picture of the dissipation being a local process. Similarly, the n-point functions of ϵ scale with exponents ρ_{2n}. The short distance singularity in Eq. (9) is an unphysical artifact of the assumed short distance scaling of the advecting velocity, mollified in real systems by viscosity.

The same method allows one to obtain information about the dissipative terms appearing in the differential equations obeyed by the structure functions and to compare our results with the early attempts [7] to calculate the anomalous exponents of the passive scalar and with the more recent ideas [8] about the behavior of the probability distribution functions in the turbulent systems.

II. INERTIAL-RANGE SCALING AND THE ZERO MODES

The correlation functions of T satisfy the (Hopf) identities which may be deduced using standard functional manipulations of stochastic differential equations, see, e.g. [9], or, for the present context, [10]. In the stationary state, the odd correlations vanish and the even ones satisfy at equal times the identities

$$\left(-\nu \sum_{j=1}^{N} \Delta_j + \tfrac{1}{2} D(d-1) \mathcal{M}_N \right) \langle T(x_1) \cdots T(x_N) \rangle$$
$$= \sum_{j<k} C(x_{jk}/L) \langle T(x_1) \underset{j}{\cdot\hat{\cdot}\cdot} \underset{k}{\cdot\hat{\cdot}\cdot} T(x_N) \rangle, \tag{10}$$

with $x_{jk} \equiv x_j - x_k$, Δ_j denoting the Laplacian in the x_j variable, and with $\mathcal{M}_N$ standing for the differential operators given by

$$\tfrac{1}{2} D(d-1) \mathcal{M}_N = -\frac{D(0)}{2} \left(\sum_{j=1}^{N} \nabla_{x_j} \right)^2$$
$$+ \tfrac{1}{2} \sum_{j \neq k} d^{\alpha\beta}(x_{jk}) \partial_{x_j^\alpha} \partial_{x_k^\beta}. \tag{11}$$

The first operator on the right-hand side of Eq. (11) is zero by translation invariance and $\mathcal{M}_N$ is a sum of the two-body operators. For $\nu>0$, the operators appearing on the left-hand side of Eqs. (10) are elliptic and positive. We may use their Green functions to solve the equations inductively. This will produce equal-time stationary correlators decaying at infinity. Physically, they describe the stationary state obtained by starting, e.g., from a fixed localized configuration of the scalar and waiting long enough.

Notice that at $\xi=0$ the operator $\mathcal{M}_N$ reduces in the translation-invariant sector to the Laplacian in N variables

x_j: $\mathcal{M}_N|_{\xi=0}=-\Delta_N=-\Sigma_{j=1}^N\Delta_j$. This implies that T becomes a Gaussian field at $\xi=0$ with the higher correlation functions built in the standard way from the two-point ones. The stationary state coincides then with that of the forced diffusion with the effective diffusion constant equal to $\nu+\frac{1}{2}D(d-1)$.

We shall describe the inertial-range correlators by taking the limit $\nu\to 0$ at fixed positions x_j and fixed large infrared cutoff L. It is not important that positions x_j be disjoint as long as we do not take derivatives of the correlators, see the remarks after Eq. (8). In the limit $\nu\to 0$, the correlation functions satisfy Eq. (10) but without the terms $\nu\Delta$. These equations completely determine the inertial-range correlators up to zero modes of operators $\mathcal{M}_N$. Physically, the zero-mode contributions are fixed by the fact that we consider the system which is the limit of the one with positive diffusivity ν. Mathematically, this means that in order to inductively solve Eqs. (10) we should use Green functions of the singular elliptic operators $\mathcal{M}_N$. Such Green functions are limits of the Green functions of the nonsingular operators corresponding to the $\nu>0$ case. It has been argued in Refs. [2–4] that the zero modes of operators $\mathcal{M}_N$ effectively appear in the inertial-range correlators and give the dominant contributions in the limit $L\to\infty$.

The zero modes in question are homogeneous under dilation, invariant by translations, rotations, parity, and symmetric under permutations of N points. Since $\mathcal{M}_N$ is a sum of two-body differential operators, zero modes of $\mathcal{M}_{N-1}$ lead by symmetrization to zero modes of $\mathcal{M}_N$. More precisely, if $f_{N-1}(x_1,\ldots,x_{N-1})$ is a zero mode of $\mathcal{M}_{N-1}$, then

$$f_N(x_1,\ldots,x_N)=\sum_{\sigma\in S_N} f_{N-1}(x_{\sigma(1)},\ldots,x_{\sigma(N-1)}), \quad (12)$$

where the sum is over the permutations of N objects, is a symmetric zero mode of $\mathcal{M}_N$. These zero modes will never contribute to the structure functions $\langle\Pi_j[T(x_j)-T(y_j)]\rangle$. At $\xi=0$, the zero modes of $\mathcal{M}_N=-\Delta_N$ are polynomials. For any even $N>2$ there is only one ''new'' zero mode of scaling dimension N that cannot be expressed as a symmetrized sum of the zero modes of $\mathcal{M}_{N-1}$. We shall denote it by E_0 (of course, E_0 is defined only up to a combination of the latter). Explicitly,

$$E_0(x_1,\ldots,x_N)=\sum_{\substack{\text{pairings }\{(l_-,l_+)\}\\ 1\leq l_-<l_+\leq N}}\prod_{(l_-,l_+)} x_{l_-l_+}^2+[\ldots] \quad (13)$$

where the dots $[\ldots]$ refer to quantities which may be written as a (symmetrized) sum of functions depending only on $N-1$ variables.

The two-point function of the scalar in the inertial range is [1,10]

$$\langle T(x_1)T(x_2)\rangle=\text{const}-\frac{2\overline{\epsilon}}{(2-\xi)Dd(d-1)}|x_{12}|^{2-\xi}+O(L^{-2}|x_{12}|^{4-\xi}), \quad (14)$$

with const $=O(L^{2-\xi})$. It follows that at $\xi=0$, where T becomes a Gaussian field,

$$\langle T(x_1)\cdots T(x_N)\rangle|_{\xi=0}\cong c_N^0E_0(x_1,\ldots,x_N)+[\ldots], \quad (15)$$

where $c_N^0=(-\overline{\epsilon}/Dd(d-1))^{N/2}$. The error not contained in the $[\ldots]$ terms is bounded by $O(L^{-2}(\max|x_{jk}|)^{N+2})$.

Upon switching on positive ξ, the symmetric zero modes of degree N will evolve to zero modes of $\mathcal{M}_N$ with a modified homogeneity. They may be found by the degenerate perturbation expansion. Again, only one of them will not come from the zero modes of $\mathcal{M}_{N-1}$. We shall call it F_0. Although for ξ positive, T is no longer a Gaussian field, its correlation functions may be inductively computed from Eq. (10). In particular, it is easy to see that the simple expressions

$$A_N\sum_{1\leq j<k\leq N}|x_{jk}|^{(2-\xi)N/2},$$

where the coefficients

$$A_N=\frac{2(N-2)!}{(N/2)!}\left(\frac{-\overline{\epsilon}}{(2-\xi)D(d-1)}\right)^{N/2}\prod_{l=0}^{N/2-1}[d+(2-\xi)l]^{-1}$$

satisfy the version of Eq. (10) with $\nu=0$ and $L=\infty$. This scaling solution obviously leads to vanishing higher structure functions and cannot give the right answer for the inertial-range correlators. The homogeneous zero modes of the operators $\mathcal{M}_N$, which enter already at the first inductive step [the constant in Eq. (14)], modify the answer. At further inductive steps, the previous step modifications will induce new ones which, however, all give rise to combinations $[\ldots]$ of functions depending on fewer variables except, eventually, for the terms proportional to zero modes of $\mathcal{M}_N$. If the homogeneity degree of the zero mode is smaller than $(2-\xi)(N/2)$, the proportionality constant will contain a compensating positive power of L and may give the contribution dominating the large L structure functions if the zero mode is not of the $[\ldots]$ type. Indeed, for small positive ξ there is only one non-$[\ldots]$-zero mode which we have denoted by F_0. Its homogeneity degree is $(2-\xi)(N/2)-\rho_N$ with positive ρ_N, as will be demonstrated below.

As a result, for small $\xi>0$,

$$\langle T(x_1)\cdots T(x_N)\rangle\cong c_NL^{\rho_N}F_0(x_1,\ldots,x_N)+[\ldots], \quad (16)$$

with the non-$[\ldots]$ error bounded by $O(L^{-2+O(\xi)}\times(\max|x_{jk}|)^{N+2+O(\xi)})$. For ξ not very small, the perturbations of zero modes which at $\xi=0$ have degree higher than N may eventually enter the interval of scaling dimensions smaller than $(2-\xi)(N/2)$ and give non-negligible or even dominant contributions to the structure functions. The large L and $\xi\to 0$ limits of the correlation functions of T do not commute since the terms scaling with different powers of L become degenerate for $\xi=0$, see [2]. These limits, however, do commute for the structure functions involving only the F_0 contribution scaling as $L^{O(\xi)}$ and the error bounded by $L^{-2+O(\xi)}$. As $F_0|_{\xi=0}=E_0$, it follows by comparison of (15) and (16) that the amplitude $c_N = c_N^0 + O(\xi)$. The $O(\xi)$ contributions to the amplitudes c_N depend on the shape of covariance C and hence are not universal.

The relation (16) implies the behavior (5) of the N-point structure functions $S_N(x) \equiv \langle [T(x) - T(0)]^N \rangle$. In the Gaussian limit,

$$S_N(x)|_{\xi=0} = a_N^0 |x|^N, \tag{17}$$

where

$$a_N^0 = \frac{N!}{(N/2)!} \left(\frac{\overline{\epsilon}}{Dd(d-1)} \right)^{N/2}.$$

It follows from the continuity of the structure functions at $\xi=0$ that the amplitude a_N in Eq. (5) is equal to $a_N^0 + O(\xi)$.

In the perturbation expansion in powers of ξ,

$$F_0 = E_0 + \xi G_0 + O(\xi^2). \tag{18}$$

In the next sections, we shall compute the $O(\xi)$ contribution G_0 (modulo [. . .] terms). Inserting the decomposition (18) into (16), we obtain an asymptotic expression for the structure functions which, although obtained by the first order zero-mode analysis, contains all orders in ξ resumming the series $\Sigma \alpha_n \xi^n (\ln L)^n$ of logarithmic infrared divergences appearing in the expansion of the structure functions in powers of ξ. This is the situation well known from the perturbative renormalization group where the first order approximation to the single renormalization group step leads upon iterations to the resummation of the leading logarithms in the perturbative expansion of correlation functions.

III. ANOMALOUS DIMENSIONS AT $O(\xi)$

Let us discuss the perturbative calculation of the homogeneous zero modes of $\mathcal{M}_N$. At the first order in ξ we have

$$\mathcal{M}_N = -\Delta_N + \xi V_N + O(\xi^2),$$

with Δ_N the Laplacian in N variables and V_N given by

$$V_N = \sum_{1 \leq j \neq k \leq N} \left(\delta^{\alpha\beta} \ln|x_{jk}| - \frac{1}{(d-1)} \frac{x_{jk}^\alpha x_{jk}^\beta}{|x_{jk}|^2} \right) \partial_{x_j^\alpha} \partial_{x_k^\beta} - \frac{1}{(d-1)} \Delta_N. \tag{19}$$

Note that, since $\mathcal{M}_N$ is a homogeneous operator of dimension $\xi - 2$, we have

$$\left[\sum x_j^\alpha \partial_{x_j^\alpha}, V_N \right] = -\Delta_N - 2V_N. \tag{20}$$

Let E be a symmetric homogeneous zero mode of $\mathcal{M}_N|_{\xi=0}$ of degree N. We shall search for the zero mode of $\mathcal{M}_N$ of the form $F = E + \xi G + O(\xi^2)$. The zero-mode equation gives at the order linear in ξ

$$-\Delta_N G + V_N E = 0. \tag{21}$$

The solutions G of this equation are clearly defined up to zero modes of Δ_N. Note that due to the scaling properties of E and V_N,

$$\begin{aligned} -\Delta_N \left(\sum x_j^\alpha \partial_{x_j^\alpha} - N \right) G &= -\left(\sum x_j^\alpha \partial_{x_j^\alpha} - N + 2 \right) \Delta_N G \\ &= -\left(\sum x_j^\alpha \partial_{x_j^\alpha} - N + 2 \right) V_N E \\ &= \Delta_N E = 0. \end{aligned} \tag{22}$$

Hence the function $E' \equiv (\Sigma x_j^\alpha \partial_{x_j^\alpha} - N) G$ is necessarily a zero mode of Δ_N. We shall show that there exist solutions G of Eq. (21) such that E' are homogeneous polynomials of degree N. Such solutions are defined up to degree N zero modes of Δ_N but this ambiguity does not show up in E'. We obtain this way a linear transformation

$$\Gamma : E \mapsto E'$$

of the space of symmetric homogeneous zero modes of Δ_N of degree N. If (E_a) is a basis of this space then the matrix (Γ_b^a) of this transformation given by $E_b' = \Gamma_b^a E_a$ plays the role of the *matrix of anomalous dimensions* at first order in ξ. Indeed, if $E = v^b E_b$ is an eigenvalue λ eigenvector of the transformation Γ, i.e., if (v^b) is an eigenvector of matrix (Γ_b^a), then, for the corresponding solution of Eq. (21), we obtain

$$\left(\sum x_j^\alpha \partial_{x_j^\alpha} - N \right) G = \lambda E \tag{23}$$

or

$$\left(\sum x_j^\alpha \partial_{x_j^\alpha} - N - \xi\lambda \right) (E + \xi G) = O(\xi^2), \tag{24}$$

which means that $E + \xi G$ is homogeneous of order $N + \xi\lambda$ up to $O(\xi^2)$. Hence the homogeneous zero modes of $\mathcal{M}_N$ are perturbations of the $\xi=0$ zero modes corresponding to eigenvectors of the matrix of anomalous dimensions. If the matrix (Γ_b^a) is not totally diagonalizable then there will be logarithmic corrections to the zero-mode homogeneity [10].

Reflecting the fact that all but one zero modes of $\mathcal{M}_N$ are obtainable from those of $\mathcal{M}_{N-1}$ by symmetrization, the matrix of anomalous dimension is block triangular. Namely,

$$(\Gamma_b^a) = \begin{pmatrix} \Gamma_0^0 & 0 \\ \Gamma_0^{a'} & \Gamma_{b'}^{a'} \end{pmatrix}$$

if the matrix is written in a basis $(E_0, (E_{a'}))$ where E_0 is the zero mode defined in Eq. (13) and $(E_{a'})$ forms a basis of the degree N zero modes arising by symmetrization of functions depending on at most $N-1$ variables. The matrix element Γ_0^0 is necessarily an eigenvalue of the matrix (Γ_b^a). If by an adequate choice of the [. . .] terms in its definition E_0 becomes the corresponding eigenvector of the transformation Γ then $F_0 = E_0 + \xi G_0 + O(\xi^2)$ describes the perturbed homogeneous zero mode of $\mathcal{M}_N$ and Γ_0^0 gives the $O(\xi)$ correction to the scaling exponent of F_0 equal to N in the leading order. The anomalous exponent of the N-point structure function is therefore given by

$$\rho_N = -\xi\left(\frac{N}{2}+\Gamma_0^0\right) \tag{25}$$

since the naive N-point scaling dimension is $(2-\xi)(N/2)$.

In principle it may happen (although it does not at least for $N=2,4,6$) that Γ_0^0 is a degenerate eigenvalue of Γ and there is no corresponding eigenvector E_0. It is easy to see, however, that even in this case there exists a zero mode $F_0=E_0+\xi G_0+O(\xi^2)$ of $\mathcal{M}_N$ which is homogeneous of degree $N+\xi\Gamma_0^0+O(\xi^2)$ up to $[\ldots]$ terms (the homogeneous terms are accompanied by the ones with powers of logarithms, the latter appearing in the $[\ldots]$ subspace). Such modifications would not affect the analysis of the structure functions.

IV. LEADING ORDER CORRECTIONS TO THE ZERO MODES

Let us return to the analysis of Eq. (21). We shall search for the solution G in the form

$$G=\sum_{j\neq k}(H_{jk}\ln|x_{jk}|)+H, \tag{26}$$

with H_{jk} and H polynomials of degree N. For such a solution,

$$E'\equiv\left(\sum x_j^\alpha\partial_{x_j^\alpha}-N\right)G=\sum_{j\neq k}H_{jk} \tag{27}$$

would necessarily be a zero mode of Δ_N of degree N, as required. We shall see that there indeed exist solutions of (21) of the form (26) (unique up to degree N zero modes of Δ_N) and that the polynomials H_{jk} scale as $|x_{jk}|^2$ when $|x_{jk}|\to 0$ assuring that the logarithms in (26) do not lead to divergent singularities in the correlation functions of T at coinciding points. Note, however, that the divergences at coinciding points start to appear in the correlation functions involving double derivatives of T or products of two first derivatives.

The substitution of the ansatz (26) into Eq. (21) gives a set of three equations for H_{jk} and H:

$$\Delta_N H_{jk}=\nabla_j\cdot\nabla_k E, \tag{28}$$

$$(d-2+x_{jk}\cdot\nabla_{jk})H_{jk}+\frac{1}{2(d-1)}(x_{jk}^\alpha x_{jk}^\beta\partial_{x_j^\alpha}\partial_{x_k^\beta})E$$
$$=-\tfrac{1}{2}x_{jk}^2K_{jk}, \tag{29}$$

$$\Delta_N H=\sum_{j\neq k}K_{jk}, \tag{30}$$

where K_{jk} are polynomials of degree $N-2$ and $\nabla_{jk}\equiv\nabla_j-\nabla_k$ with $\nabla_j=(\partial_{x_j^\alpha})$. Equation (30) is for free since any polynomial of degree $N-2$ is in the image of Δ_N acting on polynomials of degree N. Thus, given the solution H_{jk} of Eqs. (28) and (29), there always exists a degree N polynomial H solving Eq. (30) and it is unique up to the zero modes of Δ_N.

We are thus left with solving Eqs. (28) and (29). We shall first prove that there is a unique solution of these equations and then we shall produce the solution when the initial zero mode E is the ‘‘new’’ zero mode E_0 defined by Eq. (13). Notice that this is now Eq. (28) which implies that $\Sigma_{j\neq k}H_{jk}$ is a zero mode of Δ_N. By symmetry, we may specialize Eqs. (28) and (29) to $j=1,k=2$. Let us work in the variables $x=(x_1+x_2)/2$, $y=x_{12}$, and $x_3,\ldots,x_N$. We have $x_{12}\cdot\nabla_{12}=2y\cdot\nabla_y$. Since $y\cdot\nabla_y$ counts the degree in y, we shall decompose all terms entering in Eq. (29) into a sum of terms of given degree in y. Namely, $H_{12}=\Sigma^N_{p=0}H_{12}^{(p)}$, $K_{12}=\Sigma_{p=0}^{N-2}K_{12}^{(p)}$, and

$$-\frac{1}{2(d-1)}(x_{12}^\alpha x_{12}^\beta\partial_{x_1^\alpha}\partial_{x_2^\beta})E=\sum_{p=2}^N\tilde{E}^{(p)},$$

with $H_{12}^{(p)}$, $K_{12}^{(p)}$, and $\tilde{E}^{(p)}$ homogeneous polynomials in y of degree p. The operator $(d-2+2y\cdot\nabla_y)$ is invertible on such homogeneous polynomials. Equation (29) implies then that

$$H_{12}^{(0)}=H_{12}^{(1)}=0,$$

$$H_{12}^{(p)}=\frac{1}{d-2+2p}\left(\tilde{E}^{(p)}-\tfrac{1}{2}y^2K_{12}^{(p-2)}\right)\quad\text{for } 2\leqslant p\leqslant N.$$

The fact that $H_{12}^{(0)}=H_{12}^{(1)}=0$ implies that H_{12} scales as $|x_{12}|^2$ when $x_1\to x_2$. Equation (28) may then be rewritten as

$$2\Delta_yH_{12}^{(p)}=(\nabla_1\cdot\nabla_2E)^{(p-2)}-\Delta^\perp H_{12}^{(p-2)}$$

for $p\geqslant 2$, where $\Delta^\perp=\tfrac{1}{2}\Delta_x+\Sigma_{j=3}^N\Delta_j$. With the use of the previous relation between $H_{12}^{(p)}$ and $y^2K_{12}^{(p-2)}$ the latter equation takes the form

$$\Delta_y(y^2K_{12}^{(p)})=f_p$$

for some recursively known homogeneous polynomials f_p. These equations may be solved for $K_{12}^{(p)}$ since Δ_yy^2 is an invertible operator on the space of homogeneous polynomials of fixed degree.

Let us find the deformed zero mode F_0 which at $\xi=0$ reduces to E_0 of Eq. (13). To solve Eqs. (28) and (29) (by symmetry, we may again set $j=1$ and $k=2$), we first have to compute $(\nabla_1\cdot\nabla_2)E$ and $(x_{12}^\alpha x_{12}^\beta\partial_{x_1^\alpha}\partial_{x_2^\beta})E$.

$$(\nabla_1\cdot\nabla_2)E=-2(d+N-2)\sum{}'\prod_{(l_-,l_+)}x_{l_-l_+}^2+[\ldots]_{12}, \tag{31}$$

$$(x_{12}^{\alpha}x_{12}^{\beta}\partial_{x_1^\alpha}\partial_{x_2^\beta})E=-2x_{12}^2{\sum}'\prod_{(l_-,l_+)}x_{l_-l_+}^2+2\sum_{3\leqslant j<k\leqslant N}(x_{1j}^2-x_{2j}^2)(x_{1k}^2-x_{2k}^2)\times{\sum}''\prod_{(l_-,l_+)}x_{l_-l_+}^2+[\,\dots]_{12},\qquad(32)$$

where Σ' denotes the sum over pairings $\{(l_-,l_+)\}$ with $3\leqslant l_-<l_+\leqslant N$ and Σ'' a similar sum but with, additionally, $l_\pm\neq j,k$. The symbol $[\,\dots]_{12}$ refers to a sum of terms which do not depend on at least one x_p with $p\geqslant 3$. Recall from the proof of existence of solutions of Eqs. (28) and (29) that H_{12} has to scale at least as $|x_{12}|^2$ as $x_1\to x_2$. It follows then that it must be of the form

$$H_{12}=aL_{12}+bx_{12}^2{\sum}'\prod_{(l_-,l_+)}x_{l_-l_+}^2+[\,\dots]_{12}\qquad(33)$$

for some coefficients a and b where

$$L_{12}=\sum_{3\leqslant j<k\leqslant N}(x_{1j}^2-x_{2j}^2)(x_{1k}^2-x_{2k}^2){\sum}''\prod_{(l_-,l_+)}x_{l_-l_+}^2.\qquad(34)$$

Note two properties of L_{12}:

$$(x_{12}\cdot\nabla_{12})L_{12}=4L_{12},\qquad(35)$$

$$\Delta L_{12}=-4(N-2){\sum}'\prod_{(l_-,l_+)}x_{l_-l_+}^2+[\,\dots]_{12}.\qquad(36)$$

The first relation just means that L_{12} is a homogeneous function of x_{12} of degree 2. It implies that

$$(d-2+x_{12}\cdot\nabla_{12})H_{12}=(d+2)H_{12}+[\,\dots]_{12}.$$

Comparing this with relation (32), we obtain from Eq. (29) the value of the coefficient a:

$$a=-\frac{1}{(d-1)(d+2)}.$$

Next, it follows from relation (36) that

$$\Delta H_{12}=\left(\frac{4(N-2)}{(d-1)(d+2)}+4db\right){\sum}'\prod_{(l_-,l_+)}x_{l_-l_+}^2+[\,\dots]_{12}.$$

Comparison of Eqs. (28) and (31) gives the value of the coefficient b:

$$b=-\frac{1}{d}\left(\frac{N-2}{(d-1)(d+2)}+\frac{d+N-2}{2}\right).$$

This completely determines H_{12} up to terms $[\,\dots]_{12}$.

Finally, in order to find the anomalous dimension Γ_0^0, we recall that the matrix of anomalous dimensions is found by looking at $\Sigma_{j\neq k}H_{jk}$, cf. Eq. (27). Γ_0^0 is obtained by projecting relation (27) on E_0 using the triangular structure of the transformation Γ. Gathering all the terms in Eq. (33), we obtain after a simple algebra

$$\sum_{j\neq k}H_{jk}=-\frac{N(d+N)}{2(d+2)}\sum_{\substack{\text{pairings }\{(l_-,l_+)\}\\ 1\leqslant l_-<l_+\leqslant N}}\prod_{(l_-,l_+)}x_{l_-l_+}^2+[\,\dots]=-\frac{N(d+N)}{2(d+2)}E_0+[\,\dots].$$

Thus

$$\Gamma_0^0=-\frac{N(d+N)}{2(d+2)}=-\frac{N}{2}-\frac{N(N-2)}{2(d+2)},\qquad(37)$$

which via Eq. (25) leads to the claimed value (6) of the anomalous exponent ρ_N.

V. DISSIPATION FIELD

Let us sketch the argument for the equality of two definitions (7),(8) of the dissipation field $\epsilon(x)$ at $\nu=0$. First, we shall retrace the self-consistency arguments about the short distance behavior of the correlation functions [3,6]. These go as follows. By separating terms, the Hopf identity (10) may be rewritten in the form

$$\begin{aligned}&[-\nu\Delta_1-\nu\Delta_2+\tfrac{1}{2}D(d-1)\mathcal{M}_{(1,2)}]\langle T(x_1)T(x_2)T(x_3)\cdots T(x_N)\rangle\\&=\sum_{j=3}^{N}[\nu\Delta_j-\tfrac{1}{2}D(d-1)(\mathcal{M}_{(1,j)}+\mathcal{M}_{(2,j)})]\langle T(x_1)\cdots T(x_N)\rangle\\&\quad-\tfrac{1}{2}D(d-1)\sum_{3\leqslant j<k\leqslant N}\mathcal{M}_{(j,k)}\langle T(x_1)\cdots T(x_N)\rangle+\sum_{j<k}C(x_{jk}/L)\langle T(x_1)\underset{j}{\cdot\vdots\cdot}\cdots\underset{k}{\cdot\vdots\cdot}T(x_N)\rangle,\end{aligned}\qquad(38)$$

where the two-point operator $\frac{1}{2}D(d-1)\mathcal{M}_{(j,k)}\equiv d^{\alpha\beta}(x_{jk})\partial_{x_j^\alpha}\partial_{x_k^\beta}$. In variables $x=(x_1+x_2)/2$ and $y=x_{12}$ the left-hand side of Eq. (38) becomes

$$[-2\nu\Delta_y-d^{\alpha\beta}(y)\partial_{y^\alpha}\partial_{y^\beta}-\tfrac{1}{2}\nu\Delta_x+\tfrac{1}{4}d^{\alpha\beta}(y)\partial_{x^\alpha}\partial_{x^\beta}]\langle T(x_1)\cdots T(x_N)\rangle.$$

Equation (38), with the use of the latter decomposition and of the relation $\partial_{x^\alpha}=\partial_{x_1^\alpha}+\partial_{x_2^\alpha}=-\Sigma_{j=3}^{N}\partial_{x_j^\alpha}$, allows us to write

$$[-2\nu\Delta_y - d^{\alpha\beta}(y)\partial_{y^\alpha}\partial_{y^\beta}]\langle T(x+\tfrac{1}{2}y)T(x-\tfrac{1}{2}y)T(x_3)\cdots T(x_N)\rangle = R, \tag{39}$$

where R is a combination of terms involving only x_j derivatives and at most first y derivatives of $\langle T(x+\frac{1}{2}y)T(x-\frac{1}{2}y)T(x_3)\cdots T(x_N)\rangle$. Let us assume that the limit when $x_{12}\to 0$ of $\langle T(x_1)T(x_2)T(x_3)\cdots T(x_N)\rangle$ and of $\langle(\nabla T)(x_1)T(x_2)T(x_3)\cdots T(x_N)\rangle$ and of their derivatives over x_j, $j\geq 3$, exists uniformly in small $\nu>0$ (for separated $x,x_3,\ldots,x_N$). Then, as in the analysis of the two-point function (with anisotropic forcing), one infers from Eq. (39) that

$$\langle T(x+\tfrac{1}{2}y)T(x-\tfrac{1}{2}y)T(x_3)\cdots T(x_N)\rangle = c_1|y|^2[\nu+\tfrac{1}{2}D(d-1)|y|^\xi]^{-1} + \text{zero modes of } [-2\nu\Delta_y - d^{\alpha\beta}(y)\partial_{y^\alpha}\partial_{y^\beta}] + \text{error}, \tag{40}$$

the coefficients depending on $x,x_3,\ldots,x_N$ and the error more regular when $\nu\to 0$ and $y\to 0$. The zero modes contain a polynomial of the first order in y. Of the remaining zero modes the most dangerous one comes from the angular momentum 2 sector and it behaves as $O(|y|^{\alpha_2})$ for $\nu\ll\frac{1}{2}D(d-1)|y|^\xi$, where $\alpha_2=\frac{1}{2}[-d+2-\xi+\sqrt{(d-2+\xi)^2+8d}]>2-\xi$, and as $O(|y|^2)$ for $\frac{1}{2}D(d-1)|y|^\xi\ll\nu$. All such terms and their first y derivatives have limits when $y\to 0$ uniformly in small ν. As we see, our assumptions about the correlators of T are at least self-consistent. They are confirmed by our $O(\xi)$ computation of the structure functions. Indeed, at $\nu=0$ and for large L the structure functions receive the dominant contribution from the zero modes F_0 of $\mathcal{M}_N$ which behave like $\xi O(|y|^2)\ln|y|$ modulo a first order polynomial in y and $O(\xi^2)$ terms, in agreement with the above analysis. Note that the $\xi O(|y|^2)\ln|y|$ contribution to F_0 is not rotationally invariant in y: it receives contributions from both the $O(|y|^{2-\xi})$ and the $O(|y|^{\alpha_2})$ angular momentum 2 terms in $\langle T(x_1)\cdots T(x_N)\rangle$.

Let us use our self-consistent assumptions about $\langle T(x_1)\cdots T(x_2)\rangle$ in a version of Eq. (39):

$$[2\nu\nabla_1\cdot\nabla_2 + d^{\alpha\beta}(x_{12})\partial_{x_1^\alpha}\partial_{x_2^\beta}]\langle T(x_1)T(x_2)T(x_3)\cdots T(x_N)\rangle = R'. \tag{41}$$

Expression R' involves only terms with at most one derivative over x_1 or x_2. Therefore $\lim_{\nu\to 0}\lim_{x_{12}\to 0}R'$ should exist and be equal to $\lim_{x_{12}\to 0}\lim_{\nu\to 0}R'$. The same limits applied to the left-hand side of (41) give, depending on the order, the definition (7) or (8) of the dissipation field insertion $\epsilon(x)$, provided that $\xi>0$. Indeed, under first $x_{12}\to 0$ and then $\nu\to 0$ limits the $d(x_{12})\nabla_1\nabla_2$ term disappears due to vanishing of $d(x)$ at zero while sending $\nu\to 0$ before the $x_{12}\to 0$ limit kills the $\nu\nabla_1\cdot\nabla_2$ contribution. Hence the equivalence of two definitions for $0<\xi<1$.

By similar arguments, all three limits $\lim_{\nu\to 0}$, $\lim_{x_{12}\to 0}$, and $\lim_{\xi\to 0}$ commute in the action on R'. Applying them on the left-hand side of Eq. (41), we infer that at $\nu=0$

$$\lim_{\xi\to 0}\epsilon(x) = \tfrac{1}{2}D(d-1)[\nabla T(x)]^2 \tag{42}$$

and it describes the dissipation field of the scalar T diffusing with the diffusion constant $\frac{1}{2}D(d-1)$ and dissipating energy on long scales. Note that the right-hand side may be viewed as a direct application of the second definition (8) at $\xi=0$ whereas the application of the first one (7) would give $\epsilon(x)=0$: the equivalence of the definitions breaks down at $\xi=0$. At $\nu=0$, the $x'\to x$ and $\xi\to 0$ limits do not commute for $(\nabla T)(x')(\nabla T)(x)$ although they do commute for $T(x')T(x)$ or for $(\nabla T)(x')T(x)$. This is due to the disappearance of the distinction between the dissipative and the inertial-range behavior at $\xi=0$. A straightforward calculation employing the relation (42) shows that at noncoinciding points

$$\lim_{L\to\infty}\lim_{\xi\to 0}\langle\epsilon(x_1),\ldots,\epsilon(x_n)\rangle^c = 2^{n-1}(n-1)!\,d^{1-n}\overline{\epsilon^n}. \tag{43}$$

In particular, field $\epsilon(x)$ becomes constant in space at $\xi=0$ and $L=\infty$, in agreement with the physical picture of dissipation becoming a large scale phenomenon when $\xi\to 0$.

The inertial-range decay (9) follows from Eq. (16) with the use of the definition (8) of the dissipation field and of the fact that $\langle\epsilon(x)\rangle^2=\overline{\epsilon^2}$ gives for large L a subdominant contribution to $\langle\epsilon(x),\epsilon(0)\rangle^c$, see [6] for a similar analysis. From Eq. (43) we infer that the proportionality constant in (9) is equal to $2\overline{\epsilon^2}/d+O(\xi)$. Similarly, the mixed correlation functions

$$\langle\epsilon(x_1)\cdots\epsilon(x_n)T(y_1)\cdots T(y_m)\rangle$$

scale with the infrared cutoff as $L^{\rho_{2n+m}}$ and with positions with exponent $(2-\xi)(m/2)-\rho_{2n+m}$, in accordance with the fusion rule arguments of [12,13].

VI. EQUATIONS FOR STRUCTURE FUNCTIONS

Much of the past attempts to understand the behavior of the structure functions S_N was based on the differential equations satisfied by them [7]. These equations may be obtained from the N-point function equation (10) in the following way. Let $\delta_j(x,y)$ denote the difference operator acting on functions of N variables $f(x_1,\ldots,x_N)$ by subtracting their values at $x_j=x$ and $x_j=y$. $\delta_j(x,y)$ commute for different j and $\langle[T(x)-T(y)]^N\rangle=\Pi_j\delta_j(x,y)\langle T(x_1),\ldots,T(x_N)\rangle$. Application of $\Pi_j\delta_j(x,y)$ to Eq. (10) results in the identity

$$-d^{\alpha\beta}(x)\partial_{x^\alpha}\partial_{x^\beta}S_N(x)+N(N-1)\left[C\left(\frac{x}{L}\right)-C(0)\right]S_{N-2}(x) = J_N(x), \tag{44}$$

with the dissipative contribution

$$J_N(x)=2\nu N\langle(\Delta T)(x)[T(x)-T(0)]^{N-1}\rangle. \tag{45}$$

Alternatively, Eq. (44) may be obtained directly from the basic stochastic differential equation (1). Despite prefactor ν, the term J_N does not vanish when $\nu\to 0$ due to the dissipative anomaly. Indeed, coefficient ν may be absorbed into the insertions of the dissipation field:

$$J_N(x)=2\nu\Delta S_N(x)-2N(N-1)\langle\epsilon(x)[T(x)-T(0)]^{N-2}\rangle \xrightarrow[\nu\to 0]{} -2N(N-1)\langle\epsilon(x)[T(x)-T(0)]^{N-2}\rangle|_{\nu=0}. \tag{46}$$

The mutual nonsingularity of $\epsilon(x)$ and $[T(x')-T(0)]^N$ at $x=x'$ and $\nu=0$, which has been assumed in the last expression for J_N, may be checked by a self-consistent analysis or directly for the perturbative solution.

Our result (5) about the asymptotics of the structure functions and Eq. (44) imply that at $\nu=0$ and for large L

$$J_N(x)\cong -b_N(L/|x|)^{\rho_N}|x|^{(2-\xi)(N-2)/2}, \tag{47}$$

with

$$b_N=D(d-1)\left((2-\xi)\frac{N}{2}-\rho_N\right)\left(d+(2-\xi)\frac{N-2}{2}-\rho_N\right)a_N.$$

Note that the last expression may be rewritten as

$$J_N(x)\cong\frac{a_2b_N}{a_Nb_2}J_2\frac{S_N(x)}{S_2(x)}. \tag{48}$$

This relation may be confirmed by a direct calculation of J_N to order $O(\xi)$ from the dominant zero-mode F_0 contribution to the correlation functions.

In the inspiring paper [7], Kraichnan attempted to obtain anomalous exponents from Eq. (44) by assuming a relation similar to (48) but with

$$\frac{a_2b_N}{a_Nb_2}=\frac{[(2-\xi)(N/2)-\rho_N]\{d+(2-\xi)[(N-2)/2]-\rho_N\}}{(2-\xi)d}$$

replaced by $N/2$. This was further argued for in [6]. Kraichnan's assumption led upon insertion into (44) to the quadratic equation for the scaling dimension $\zeta_N\equiv(2-\xi)\times(N/2)-\rho_N$,

$$\zeta_N(\zeta_N+d-2+\xi)=(2-\xi)d\frac{N}{2},$$

whose solution gave the anomalous exponents ρ_N. Note that the replacement of the factor $N/2$ on the right-hand side of Kraichnan's equation for ζ_N by a_2b_N/a_Nb_2 leads instead to a tautological identity.

Kraichnan's values of ρ_N, unlike the ones obtained in the present work, do not vanish at $\xi=0$. The latter might seem strange in view of the fact that T becomes a Gaussian field at $\xi=0$ and the ansatz of [7] was fit with the Gaussian calculation. The latter was based, however, on the definition (45) applied directly in the Gaussian case whereas at $\xi=0$ one should use for J_N the expression (46) with $\epsilon(x)$ given by Eq. (42). The latter calculation agrees, of course, with Eq. (48):

$$J_N(x)|_{\xi=0}=-b_N^0|x|^{N-2}, \tag{49}$$

where $b_N^0=D(d-1)N(d+N-2)a_N^0$ see Eq. (17). In this limit, the differential equation (44) reduces to $-D(d-1)\Delta S_N|_{\xi=0}=J_N|_{\xi=0}$.

Above, we have studied the inertial-range behavior of the structure functions. More general objects to study are the joint probability distribution functions (PDF's) $P_N(T_1,\ldots,T_N;\ x_1,\ldots,x_N)$ of the scalar whose moments give the equal-time correlation functions. In particular, the structure functions $\langle\Pi_j[T(x_j)-T(y_j)]\rangle$ are special moments of the PDF's

$$Q_N(T_1,\ldots,T_N;x_1,\ldots,x_N)=\int P_N(T_1+\tau,\ldots,T_N+\tau;x_1,\ldots,x_N)d\tau, \tag{50}$$

which are translational invariant in the T variables. The generating function of S_N's,

$$Z(\lambda;x)\equiv\langle e^{i\lambda[T(x)-T(0)]}\rangle=\int e^{i\lambda T}Q(T;x)dT \tag{51}$$

is a Fourier transform of the PDF $Q(T_1-T_2;x_{12})\equiv Q_2(T_1,T_2;x_1,x_2)$.

In a recent paper [8] on the Burgers equation, Polyakov has argued that the structure-function PDF's exhibit a universal inertial-range behavior for (translating his statements to the passive scalar case) $T_i\ll T_{\rm rms}$ where $T_{\rm rms}\equiv\sqrt{\langle T(0)^2\rangle}=O(L^{1-\xi/2})$, see Eq. (14). Polyakov's analysis was based on postulating an operator product expansion which allows us to close the resummed version of Eq. (44),

$$-d^{\alpha\beta}(x)\partial_{x^\alpha}\partial_{x^\beta}Z(\lambda;x)+\lambda^2\left[C(0)-C\left(\frac{x}{L}\right)\right]Z(\lambda;x)=J(\lambda;x), \tag{52}$$

by expressing its right-hand side $J(\lambda;x)\equiv\Sigma_N[(i\lambda)^N/N!]J_N(x)=2\lambda^2\langle\epsilon(x)e^{i\lambda[T(x)-T(0)]}\rangle$ again in terms of $Z(\lambda;x)$. The resulting equation for Z may be reduced to an ordinary differential equation by imposing the scaling relation

$$[2x\cdot\nabla-(2-\xi)\lambda\partial_\lambda]Z(\lambda;x)=0, \tag{53}$$

i.e., by postulating that $Z(\lambda;x)$ depends on $\lambda^2|x|^{2-\xi}$. Note that a strictly scaling solution for $Z(\lambda;x)$ implies either the Kolmogorov scaling or divergence of the structure functions. For example, the resummation of the expressions (17) for the $\xi=0,\nu=0,L=\infty$ structure functions gives the Gaussian generating function

$$Z(\lambda;x)|_{\xi=0}=\exp\left[-\frac{\overline{\epsilon}}{Dd(d-1)}\lambda^2|x|^2\right], \tag{54}$$

which scales in accordance with the normal scaling of the $\xi=0$ structure functions. It corresponds to the Gaussian PDF. A straightforward check shows that at $\xi=0$ the dissipative term J may be simply expressed in terms of Z itself: $J=2\overline{\epsilon\lambda}^2[1+(1/d)\lambda\partial_\lambda]Z$.

Our small ξ analysis of the passive scalar does not allow us to prove or disprove Polyakov's picture since the individual structure functions that we study probe the small λ behavior of Z, i.e., the large T behavior of $Q(T)$. What follows from it is that the large T tails of the PDF's violate scaling for $\xi>0$. In particular, the solution (5), (6) for S_N's leads to the relation

$$\left[2x\cdot\nabla-\left(2-\frac{\xi d}{d+2}\right)\lambda\partial_\lambda+\frac{\xi}{d+2}(\lambda\partial_\lambda)^2\right]Z(\lambda;x)=O(\xi^2). \tag{55}$$

It would be interesting to recover the tail of the structure-function PDF Q in the order $O(\xi)$. This would require the knowledge of the $O(\xi)$ contributions to the nonuniversal amplitudes a_N in the relation (5) which we have not computed. It is possible that they may be found by a perturbative analysis of instanton contributions to the functional integral for the dynamical scalar correlators [11]. We expect that the refined perturbative analysis in ξ will allow a better control of both the structure functions and the corresponding PDF's.

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